SALEM–SCHAEFFER MEASURES OF DYNAMICAL SYSTEM ORIGIN

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Abstract. A class of measure preserving $\mathbb{Z}^d$- and $\mathbb{R}^d$-actions T is constructed possessing the following properties: the spectrum of T is simple and for a dense set of functions f the spectral measures σ_f have an extremal rate of the Fourier coefficient decay:

$$\widehat{\sigma}_f(n) = O(|n|^{-d/2+\varepsilon})$$

for any $\varepsilon > 0$, where the exponent $-d/2$ is the minimal possible for singular mesures on $\mathbb{T}^d$.

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1. Constructions of singular Borel measures on $[0,1]$

In this work we construct a new class of dynamical systems with simple spectrum generating spectral measures characterized by fast Fourier coefficient decay. We start with a well-known construction due to Riesz [16]. He proposed to consider a formal infinite product

$$\prod_{n=1}^{\infty} \bigl(1 + a_n \cos(\omega_n x + \phi_n)\bigr), \tag{1}$$

where $\omega_n \in 2\pi\mathbb{Z}$ is an increasing sequence, $0 < a_n \le 1$ and $\phi_n \in \mathbb{R}$. It is well known that for a certain choice of parameters a_n, ω_n and ϕ_n, for example, if $\omega_{n+1}/\omega_n \ge q > 3$ and $\sum_n a_n^2 = \infty$, this product represents a singular measure on $[0,1]$ (see [26], § 7). We understand this statement as follows. The finite products

$$\rho_N(x) = \prod_{n \le N} \bigl(1 + a_n \cos(\omega_n x + \phi_n)\bigr)$$

are interpreted as densities of probability measures on $[0,1]$, and we have convergence

$$\rho_N(x)\, ds \to d\sigma \qquad n \to \infty$$

in the weak topology, where σ is a measure on $[0,1]$. The infinite products (1) today refered to as *classical Riesz products* as well as generalized Riesz products

$$\mathscr{P} = \prod_{n=1}^{\infty} P_n(z), \quad \text{where} \quad P_n(z) = \sum_{k=0}^{q_n-1} c_{n,k}\, z^k, \quad z \in \mathbb{C}, \quad |z| = 1,$$

provide an important construction of singular measures broadly applied in analysis and dynamical systems (see [1], [2], [4], [5], [21]).

Let us denote $\mathscr{M}([0,1])$ or simply $\mathscr{M}$ the class of all Borel probability measures on $[0,1]$. A measure ν is *absolutely continuous* with respect to another measure μ, notation: $\nu \ll \mu$,

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if $\nu = p(x)\mu$, where $p(x)$ is a certain density, $p(x) \in L^1(\mu)$. The relation $\nu \ll \mu$ is a partial order on $\mathscr{M}$. The measures μ and ν in the class $\mathscr{M}$ are called *mutual singular*, $\mu \perp \nu$, if there exists a Borel set E such that $\mu(E) = \nu(E^c) = 1$, where E^c is the complement to the set E. Recall that any Borel measure σ on a segment in the real line is uniquely expanded in a sum

$$\sigma = \sigma_d + \sigma_s + \sigma_{ac}, \qquad \sigma_s \perp \lambda, \quad \sigma_{ac} = p(x)\lambda, \tag{2}$$

of discrete (a sum of atoms), purely singular and absolutely continuous components, where λ is the normalized Lebesgue measure on the segment. Since every measure $\sigma \in \mathscr{M}([0,1))$ can be considered as a measure on $\mathbb{R}$ one can define the *Fourier transform* of σ by the formula

$$\mathsf{F}[\sigma](t) = \int_0^{2\pi} e^{-2\pi i\, tx}\, d\sigma(x), \qquad t \in \mathbb{R}.$$

For probability measures on $\mathbb{R}$ the following general observation holds (see [22], ch. 2, § 12). If the Fourier transform of σ belongs to $L^1(\mathbb{R})$, for example, if $\mathsf{F}[\sigma](t) = O(t^{-1-\alpha})$, $\alpha > 0$, then σ in absolutely continuous. At the same time, any measure $\sigma \in \mathscr{M}([0,1))$ as a measure on the compact group $\mathbb{T} = \mathbb{R}/\mathbb{Z} \simeq [0,1)$ generates the sequence of *Fourier coefficients* $\widehat{\sigma}(n)$ supported on the dual group $\widehat{\mathbb{T}} = \mathbb{Z}$. Note that

$$\widehat{\sigma}(n) = \mathsf{F}[\sigma](n) = \int_0^{2\pi} e^{-2\pi i\, nx}\, d\sigma(x), \qquad n \in \mathbb{Z}.$$

Suppose that $\widehat{\sigma} \in l^2(\mathbb{Z})$. Then the Fourier series $\sum_n \widehat{\sigma}(n)\, e^{2\pi i\, nt}$ converges in the space $L^2(\mathbb{T})$ to some function $p(x)$. Using Cauchy–Schwarz inequality $\|p\|_1 = \langle |p|,\, 1\rangle \le \|p\|_2$ we see that $p(x)$ is a density of some measure $p(x)\,dx \in \mathscr{M}(\mathbb{R})$, and $\widehat{p}(n) = \widehat{\sigma}(n)$ (eg. see [25], § 1). Furhter, notice that any sequence $c_n = O(n^{-1/2-\alpha})$, $\alpha > 0$, is square summable.

In the case of singular measure σ, as a rule, we deal with a divergent series $\sum_n \widehat{\sigma}(n)\, z^n$. Understanding analytic properties of a singular measure σ, when we know certain combinatorial properties of the sequence $\widehat{\sigma}(n)$, becomes a very complicated problem. Louzin [9] constructed the first example of power series $\sum_n c_n\, z^n$ with $c_n \to 0$ divergent everywhere on the unit circle $|z| = 1$. Further, Neder [13] proved that any series $\sum_n c_n\, z^n$ with the property $\sum_n |c_n|^2 = \infty$ can be transformed to everywhere divergent (for $|z| = 1$) using some phase correction $\tilde{c}_n = e^{i\,\phi_n}\, c_n$. Now let us turn to expansion (2) of σ and remark that Riemann–Lebesgue lemma can be interperted in the folowing way.

Lemma 1. *Given an absolutely continuous measure σ_{ac},*

$$\widehat{\sigma}_{ac}(n) \to 0 \quad \textit{as} \quad n \to \infty.$$

Definition 2. We call *Menshov–Rajchman measure*, a singular measure satisfying $\widehat{\mu}(n) \to 0$, $n \to \infty$. We denote as $\mathscr{R}$ the class of all measures of such kind.

Evidently any discrete measure σ_d is never of Menshov–Rajchman type since its Fourier transform $\widehat{\sigma}_d(n)$ is a Bohr almost periodic sequence. At the same time, it is easy to see that the singular Cantor–Lebesgue measure μ_{CL} supported on the standard 1/3-Cantor set enjoys the property $\widehat{\mu}_{\mathrm{CL}}(3^k n) = \widehat{\mu}_{\mathrm{CL}}(n)$, which is explained by fractal symmetry of the Cantor set. Thus, $\mu_{\mathrm{CL}} \notin \mathscr{R}$. Modifying the construction of μ_{CL} Menshov [11] provided the first example of singular measure in the class $\mathscr{R}$. Further, Neder [12] proved that any Menshov–Rajchman measure cannot be a mixture of discrete and continuous component, and then Wiener [24] extended this result and showed that the Fourier coefficients of any continuous measure μ

converge to zero in average, and any set $\{n\colon\ |\widehat{\mu}(n)| > b > 0\}$ has zero density in $\mathbb{Z}$. Littlewood [8] found a singular probability measure σ with the rate of decay

$$\widehat{\sigma}(n) = O\big(|n|^{-c}\big), \qquad c > 0.$$

Then Wiener and Wintner [25] obtained a stronger result demonstrating that the exponent c can be arbitrary close to $1/2$, but the approach proposed by the authors generates a measure σ that depends on $c = 1/2 - \alpha$, $\alpha > 0$. Soon after this work Schaeffer [21] using the idea of Riesz products proved the existence of a singular σ with

$$\widehat{\sigma}(n) = O\big(r(|n|) \cdot |n|^{-1/2}\big)$$

for any given increasing sequence $r(n) \to \infty$, $n \to \infty$. In particular, σ satisies

$$\widehat{\sigma}(n) = O(|n|^{-1/2+\varepsilon}) \quad \text{for any} \quad \varepsilon > 0.$$

Ivashev-Musatov [6] got a further improvement of Schaeffer's result. He found a set of singular measures with sub-$|n|^{-1/2}$ rate of correlation decay satisfying $\widehat{\sigma}(n) = O(\rho(n) \cdot |n|^{-1/2+\varepsilon})$ with $\rho(n) \to 0$ but $\rho(n) \gg |n|^{-\varepsilon}$ for any $\varepsilon > 0$. Following [25] let us denote $\kappa(\sigma)$ the infinum of real γ's such that $\widehat{\sigma}(n) = O(|n|^{\gamma})$. In a series of works [18, 19, 20] Salem introduced an approach that helps to see, in particular, explicit examples of singular distributions with the property $\kappa(\sigma) = -1/2$.

Definition 3. Let us call singular measures on $\mathbb{T}^d$ (respectively, $\mathbb{R}^d$) satisfying the condition $\widehat{\sigma}(n) = O(|n|^{-d/2+\varepsilon})$ for any $\varepsilon > 0$, *measures of Salem–Schaeffer type.*

In this note we discover that Salem–Schaeffer measures appear as spectral measures for a class of group actions with invariant measure. Let us consider a measure preserving invertible transformation $T\colon\ X \to X$ of the standard Lebesgue space $(X, \mathcal{B}, \mu)$ and define *Koopman operator* on the space $H = L^2(X, \mathcal{B}, \mu)$,

$$\hat{T}\colon\ H \to H\colon\ f(x) \mapsto f(Tx).$$

Clearly, $\hat{T}$ is a unitary operator in a separable Hilbert space H, hence, it is characterized up to a unitary equivalence by the pair $(\sigma_{(T)}, \mathcal{M}(z))$, where $\sigma_{(T)}$ is the *measure of maximal spectral type* and $\mathcal{M}(z)$ is the *multiplicity function.* Of course, two transformations which are spectrally isomorphic, need not be isomorphic as dynamical systems. For example, all Bernoulli shifts have Lebesgue spectrum of infinite multiplicity but they are dinstiguished by entropy. And, in fact, it is a hard problem far from complete understanding to classify all pairs $(\sigma_{(T)}, \mathcal{M}(z))$ that can appear as spectral invariants of a measure preserving transformation (see [7, 10]). Further, given an element $f \in L^2(X, \mathcal{B}, \mu)$, the *spectral measure* $\sigma_f \in \mathscr{M}$ on $\mathbb{T} \simeq [0, 1)$ is uniquely defined by the relation

$$\widehat{\sigma}_f(n) = \int e^{-2\pi i\, xn}\, d\sigma(x) = R_f(-n) \overset{def}{=} \left\langle \hat{T}^{-n} f,\, f \right\rangle.$$

It is easy to see that $\sigma_f \ll \sigma_{(T)}$.

2. Dynamical systems generating Salem–Schaeffer measures

In this section we define actions of the groups $\mathbb{Z}^d$ and $\mathbb{R}^d$ with invariant measure generating spectral measures of Salem–Schaeffer type for a dense set of function on the phase space. Without loss of generality we concentrate our attention on the case of $\mathbb{Z}^d$-actions.

2.1. **Main construction.** Consider a nested sequence of lattices Γ_n in the group $G = \mathbb{Z}^d$ such that $G_{n+1} \subset G_n$, and let $M_n = G/\Gamma_n$ be the corresponding sequence of homogeneous spaces linked by a natural projection $\pi_n\colon\ M_{n+1} \to M_n$ mapping $a + \Gamma_{n+1}$ onto the point $a + \Gamma_n$. Let us also fix a Fölner sequence of Γ_n-fundamental domains U_n, such that $\lambda(\partial U_n) = 0$, where λ is the Haar measure on G (the condition automatically holds for $G = \mathbb{Z}$). To simplify understanding of the construction let us consider a particular case $\Gamma_n = h_n\mathbb{Z}^d$, $h_{n+1} = q_n h_n$, $q_n \in \mathbb{Z}$, and let U_n be rectangles $U_n = [0, h_n)^{\times d}$. Let us then introduce a family of maps

$$\phi_n\colon\ M_{n+1} \to M_n,$$
$$\phi_n(\gamma + u) = u + \alpha_{n,\gamma}, \qquad \gamma \in \Gamma_n/\Gamma_{n+1}, \quad u \in U_n, \alpha_{n,\gamma} \in G.$$

The sequence of maps ϕ_n difines a dynamical system on a projective limit of the spaces M_n. Thus, the parameters of our construction are: (a) the group G; (b) a sequence of lattices Γ_n; (c) rotation parameters $\alpha_{n,\gamma}$. To simplify the construction suppose that

$$\Gamma_n = h_n\mathbb{Z}^d, \qquad h_{n+1} = q_n h_n, \qquad q_n \in \mathbb{N}, \qquad q_{n+1} > 2q_n, \qquad U_n = [0, h_n)^{\times d}.$$

We represent M_{n+1} as a finite union of domains $\gamma + U_n$, where $\gamma \in \Gamma_n/\Gamma_{n+1}$. Observe that each domain U_n projects one-to-one onto M_n (up to a null set) and then rotated by $\alpha_{n,\gamma}$.

Let X be the inverse limit of the spaces M_n,

$$X = \big\{x = (x_1, x_2, \dots, x_n, \dots)\colon\ x_n \in M_n,\ \phi_n(x_{n+1}) = x_n\big\}.$$

The space X is endowed with the Tikhonov topology and a structure of probability space $(X, \mathcal{B}, \mu)$, where μ is a Borel measure that projects to the unique invariant measure μ_n on each space M_n. Finally let us define an action of the group G on the space $(X, \mathcal{B}, \mu)$. Given $t \in G$ and using Borel–Cantelli lemma we see that the probability

$$\mu\big\{x \in X\colon\ \exists\, n_0(x) \quad \forall\, n \geq n_0(x) \quad \boldsymbol{\gamma}_n(x_n) = \boldsymbol{\gamma}_n(t + x_n)\big\} = 1,$$

hence, for the points x of such kind (belonging to the set above) we can define T^t by the rule $(T^t x)_n = t + x_n$ for $n \geq n_0(x)$. For indexes $n < n_0(x)$ the coordinates $(T^t x)_n$ are recovered using the fundamental equation $\phi_n(x_{n+1}) = x_n$. The following lemma directly follows from the definition (see also [15] for the careful examination of the case $G = \mathbb{Z}$).

Lemma 4. *The maps T^t giges a measure preserving G-action on the space $(X, \mathcal{B}, \mu)$.*

We call the constructed class of G-actions *systems of iceberg type*. It can be easilily seen that our construction naturally extends a general (C, F) construction of rank one actions of Lie groups (see [3, 14]).

Let us also observe that the $\mathbb{Z}$-action of iceberg type with $q_n = 2$, $(\alpha_{n,0}, \alpha_{n,1}) = (0, h_n/2)$, is identical to the classical Morse transformation (see [23] for the discussion of arithmetic properties of Morse systems). Actions with infinite invariant measure possessing fast correlation decay and simple spetrum was studied by Ryzhikov [17].

3. Spectral properties

Theorem 5. *Let $\alpha_{n,\gamma}$ be a family of independent random variables, uniformely distributed on finite sets Γ_n/Γ_{n+1}. Then for a certain sequence $q_n \to \infty$ and for a set of (cylindric) functions f dense in $L^2(X, \mu)$ the spectral measures σ_f satisfy almost surely the condition*

$$\widehat{\sigma}_f(t) = O(|t|^{-d/2+\varepsilon}) \tag{3}$$

for any $\varepsilon > 0$. In particular, $\widehat{\sigma}_f \notin L^2(G)$, and $\widehat{\sigma}_f \in L^p(G)$ for $p > 2$.

In the case $G = \mathbb{Z}$ the proof of the next theorem is given in [15]. The next theorem, which proof goes out of this paper explains the interest to dynamical system constructions of Salem–Schaeffer measures.

Theorem 6. *There exist actions of iceberg type having pure singular spectrum falling into the class of Salem–Schaeffer measures.*

Proof of theorem 5. We establish for the constructed class of systems the following universal estimate

$$\mathsf{E}_{\mathsf{P}}|R_f(t)|^2 \le r(n)\cdot t^{-d}$$

where $h_{n-1} \le \|t\| \le h_n$, and $r(n)$ is a slowly increasing function, $r(n) \ll h_n^c$ for any $c > 0$. Here $\mathsf{E}_{\mathsf{P}}\xi$ denotes the expection of a random variable ξ according to the probability measure in the space of parameters. Consider a bounded cylindric function f with zero mean that depends only on the coordinate x_{n_0}, namely, $f(x) = f_{n_0}(x_{n_0})$. To illustrate the method of the proof let us first calculate the expectation $\mathsf{E}_{\mathsf{P}}R_f(t)$ for sufficiently large t, and without loss of generality consider $t = h_{n-1}\cdot s \in \Gamma_{n-1}$, $s \in \mathbb{Z}^d \setminus \{0\}$, $\|t\| \ll h_n$. In this case the translation by t preserves the partition of the space M_n into domains $\gamma + U_{n-1}$. Given such t the function $R_f(t)$ is approximated by the following correlation function

$$R_n^\circ(t) = h_n^{-d}\int_{M_n} f_n(x-t)\,\overline{f_n(x)}\,d\lambda_n = \mathsf{E}_{\mu_n}\left\langle f_n(x-t),\, f_n(x)\right\rangle,$$

where λ_n is the standard Haar measure on $G = \mathbb{Z}^d$, further, $\mu_n = h_n^{-d}\lambda_n$ is an invariant probability measure on M_n, and f_n is the lift of the function f to the manifold M_n. Then

$$\mathsf{E}R_n^\circ(t) = \frac{1}{Q_{n-1}}\sum_{\gamma\in\Gamma_{n-1}/\Gamma_n}\mathsf{E}\left\langle \rho_{\alpha_{n-1,\gamma}}f_{n-1},\, \rho_{\alpha_{n-1,\gamma+s}}f_{n-1}\right\rangle = 0,$$

$$Q_{n-1} = q_{n-1}^d = \#\Gamma_{n-1}/\Gamma_n,$$

since $\alpha_{n-1,\gamma}$ are $\alpha_{n-1,\gamma+s}$ are independent and $\int_X f\,d\mu = 0$. Developping this technique let further study the expectation $\mathsf{E}_{\mathsf{P}}|R_f(t)|^2$. Assume again that $t \in \Gamma_{n-1}$. By analogy approximating $R_f(t)$ by $R_n^\circ(t)$ we get

$$|R_n^\circ(t)|^2 = \frac{1}{Q_{n-1}}\sum_{\gamma_1,\gamma_2} f_{n-1}(t-\alpha_{n-1,\gamma_1})\, f_{n-1}(t-\alpha_{n-1,\gamma_1+s})\, \bar f_{n-1}(t-\alpha_{n-1,\gamma_2})\, \bar f_{n-1}(t-\alpha_{n-1,\gamma_2+s})$$

Each term is this sum is determined by four indexes $\gamma_1,\ \gamma_1+s,\ \gamma_2,\ \gamma_2+$. If any of these indexes becomes *free*, that is never repeated in the product (does not coincide with the remaining three indexes), then the expectation of the product will be zero be the independence of the random variables $\alpha_{n-1,\gamma}$. Thus, each non-trivial term in the sum participate in one of the following configurations: either $\gamma_1 = \gamma_2$ or $\gamma_2 = \gamma_1 + s \pmod{h_{n-1}}$ and simultaneously $\gamma_1 = \gamma_2 + s \pmod{h_{n-1}}$. The second configuration is statistically rare and can be neglected. The main influence in the sum is given by the first type of configurations, and we have

$$\mathsf{E}_{\mathsf{P}}|R_n^\circ(t)|^2 = \frac{1}{Q_{n-1}^2}\sum_{\gamma}\mathsf{E}_{\mathsf{P}}|f_{n-1}(t-\alpha_{n-1,\gamma})|^2\mathsf{E}_{\mathsf{P}}|f_{n-1}(t-\alpha_{n-1,\gamma+s})|^2\cdot(1+o(1)) \sim$$

$$\sim \frac{1}{Q_{n-1}}\,\mathsf{E}_{\mu_{n-1}}\mathsf{E}_{\mathsf{P}}|R_{n-1}^\circ(t)|^2.$$

Analogously one can check that the same estimate is true for $t \notin \Gamma_{n-1}$. Thus,

$$\mathsf{E}_\mathsf{P}\mathsf{E}_{\mu_n}\|R_n^\circ\|^2 \le \mathsf{E}_\mathsf{P}\mathsf{E}_{\mu_{n-1}}\|R_{n-1}^\circ\|^2 + h_n^d \cdot h_{n-1}^{-d}\mathsf{E}_\mathsf{P}\mathsf{E}_{\mu_{n-1}}\|R_{n-1}^\circ\|^2 \le \\ \le 2\,\mathsf{E}_\mathsf{P}\mathsf{E}_{\mu_{n-1}}\|R_{n-1}^\circ\|^2 \cdot (1+o(1)) = O((2+\varepsilon_0)^n), \qquad \varepsilon_0 > 0,$$

hence,

$$\mathsf{E}_\mathsf{P}|R_f(t)|^2 \sim \mathsf{E}_\mathsf{P}|R_n^\circ(t)|^2 = O(t^{-d}\cdot(2+\varepsilon_0)^n), \qquad t \ll h_n,$$

and for some slowly increasing function $\rho(t)$ such that $\rho(t) = o(h_n^c)$ for any $c > 0$ and $h_{n-1} \le \|t\| \le h_n$, we have $|R_f(t)| \le \sqrt{\rho(t)}\cdot t^{-d/2}$, and the proof is finished. □

It follows directly from theorem 5 that $\sigma_f * \sigma_f \ll \lambda$, where λ is the normalized invariant measure on $\widehat{G}$. Thus, our observation is connected to the open question due to Banach, — *"Does there exist a $\mathbb{Z}$-action with invariant probability measure having Lebesgue spectrum of multiplicity one?"* — since the spectral multiplicity for almost every action in theorem 5 equals one (and for *all* systems of iceberg type we have $\mathcal{M}(z) \le 4$).

Open questions and hypotheses.

(i) Is it true that dynamical systems in theorem 5 have singular spectrum almost surely

(ii) Can we find a speed of Fourier coefficient decay $\widehat{\sigma}(t) = O(|t|^{-1/2}\,\rho(t))$, that can be reached in the class of *all* singular measures, but *impossible* for measures of maximal spectral type generated by measure preserving transformations?

(iii) Given a measure of Salem–Schaeffer type, $\kappa(\sigma) = -1/2$, is it posible to reach any speed of decay of type

$$\widehat{\sigma}(t) = O(|t|^{-1/2}\,\rho(t)) \quad \text{with} \quad \forall\,\varepsilon > 0 \quad |t|^{-\varepsilon} \le \rho(t) \le |t|^{\varepsilon}$$

just multiplying by some density $p(x) \in L^1(\sigma)$?

(v) Is it true that all $\mathbb{Z}^d$- and $\mathbb{R}^d$-actions of iceberg type have singular spectrum?

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